Fuzzy Logic for the Islamic Knowledge

Amin A.K.Vazir*

1. CRISP RELATION VERSUS FUZZY RELATION

Elements within a set or elements in different sets often have special connections with one another. These connections are actually relationships that we dealt with everyday and that occur in many context.

These relationships in mathematical term are known as "relation". e.g. the relationships between a business and its telephone number, an employee and his or her salary, a person and a relative and so on. A relation can be considered as a set of tuples, where a tuple is an ordered pair. A binary tuple is denoted as (x, y), a ternary tuple as (x, y, z) and in general an n-ary tuple as $(x_1, x_2, x_3 \ldots x_n)$. As an example, let X = {a, b, c} and Y = {d, e, f, g, h, i, j}, then the relation (binary relation) 'father of' on $X \times Y$ may be {(a, d), (a, e), (b, f), (b, g), (b, h), (b, i), (c, j)}.

Let the relation be a subset of $X_1 \times X_2 \times X_3 \times \ldots \times X_n$, then the characteristics function of the relation R assigns a value $\mu_R(x)$ to every x in the domain set $X_1 \times X_2 \times X_3 \times \ldots \times X_n$, such that

$$\mu_R(x) = \begin{cases} 1 & \text{for } x \in R \\ 0 & \text{for } x \notin R \end{cases}$$

i.e. $\mu_R: X_1 \times X_2 \times X_3 \times \ldots \times X_n \rightarrow \{0, 1\}$

Just as the characteristic function of a crisp set can be generalized to allow for degrees of set membership, the characteristic function of a crisp relation can be generalized to allow tuples to have degrees of membership within the relation. Thus, a fuzzy relation is a fuzzy set defined on the Cartesian product of crisp sets $X_1, X_2, X_3, \ldots, X_n$, where tuples $(x_1, x_2, x_3, \ldots, x_n)$ may have varying degrees of membership within the relation. The grade of membership indicates the strength of the relation present between the elements of the tuple.

* Asstt. Prof. (Computer Science), Sheikh Zayed Islamic Centre, University of Karachi.

Let the relation R be a subset of the Cartesian product $X_1 \times X_2 \times X_3 \times \ldots \times X_n$ then the relationship function of the relation R assigns a value $\mu_R(x)$ to every x in the product $X_1 \times X_2 \times X_3 \times \ldots \times X_n$ such that

$$\mu_R(x): X_1 \times X_2 \times X_3 \times \ldots \times X_n \rightarrow [0, 1]$$

Let X = 1, 2, 3, then 'approximately equal' is the binary fuzzy relation

1/(1,1) + 1/(2,2) + 1/(3,3) + 0.8/(1,2) + 0.8/(2,3) + 0.8/(2,1) + 0.8/(2,1) + 0.8/(3,2) + 0.3/(1,2) + 0.8/(0.3,1)

The membership relation μ_R of this relation can be described by

$$\mu_R(x, y) = \begin{cases} 1 \text{ for } x = y \\ 0.8 \text{ for } |x - y| = 1 \\ 0.3 \text{ for } |x - y| = 2 \end{cases}$$

In matrix notation this can be expressed as

$$X = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 1 & 0.8 & 0.3 \\ 2 & 0.8 & 1 & 0.8 \\ 3 & 0.3 & 0.8 & 1 \end{array}$$

Take another example

Let X = {Makkah, Mina} and

Y = {Arafah, Makkah, Muzdalifah}

Let R be a fuzzy relation "far". This relation can be written as

$\mu_R(x, y)$ = 1/(Makkah, Arafah) + 0/(Makkah, Makkah) + 0.5/(Makkah, Muzdalifah) + 0.75/(Mina, Arafah) + 0.25/(Mina, Makkah) + 0.25/(Mina, Muzdalifah)

which can be expressed in matrix notation as

	Arafah	Makkah	Muzdalifah
Makkah	1	0	0.5
Mina	0.75	0.25	0.25

2. OPERATIONS ON FUZZY RELATIONS

In this section, we will deal with some important operations on fuzzy relations like intersection, union, projection, and cylindrical extension and composition. These relations are very important in

fuzzy logic because they can describe interaction between variables. This is particularly interesting in if-then rules.

2.1. Fuzzy Intersection

Let R and S are binary relations defined on X × Y. The intersection of R and S is defined by

$$\forall\ (x, y) \in X \times Y: \mu_{R \cap S}(x, y) = \min(\mu_R(x, y), \mu_S(x, y))$$

Example

Let R = "x is more muttaqi than y":

	y_1	y_2	y_3	y_4
x_1	0.8	1	0.1	0.7
x_2	0	0.8	0	0
x_3	0.9	1	0.7	0.8

and S = "x is closer to Allāh than y":

	y_1	y_2	y_3	y_4
x_1	0.4	0	0.9	0.6
x_2	0.9	0.4	0.5	0.7
x_3	0.3	0	0.8	0.5

then R ∩ S =

	y_1	y_2	y_3	y_4
x_1	0.4	0	0.1	0.6
x_2	0	0.4	0	0
x_3	0.3	0	0.7	0.5

Now if we use the formula,

$$\frac{\mu_R(x, y) \,.\, \mu_S(x, y)}{\mu_R(x, y) + \mu_S(x, y) \;-\; \mu_R(x, y) \,.\, \mu_S(x, y)}$$

then the result is

R ∩ S =

	y_1	y_2	y_3	y_4
x_1	0.3636	0	0.0989	0.4773
x_2	0	0.3636	0	0
x_3	0.2903	0	0.5957	0.4444

2.2. Fuzzy Union

Let R and S are binary relations defined on X × Y. The union R and S is defined by

$$\forall (x, y) \in X \times Y: \mu_{R \cup S}(x, y) = \max(\mu_R(x, y), \mu_S(x, y))$$

Example

Consider R and S once again

Let R = "x is more muttaqi than y":

	y_1	y_2	y_3	y_4
x_1	0.8	1	0.1	0.7
x_2	0	0.8	0	0
x_3	0.9	1	0.7	0.8

and S = "x is closer to Allāh than y":

	y_1	y_2	y_3	y_4
x_1	0.4	0	0.9	0.6
x_2	0.9	0.4	0.5	0.7
x_3	0.3	0	0.8	0.5

then R ∪ S =

	y_1	y_2	y_3	y_4
x_1	0.4	0	0.1	0.6
x_2	0	0.4	0	0
x_3	0.3	0	0.7	0.5

Now if we use the formula, $\mu_R(x, y) + \mu_S(x, y) - \mu_R(x, y) . \mu_S(x, y)$, then the result is

R ∪ S =

	y_1	y_2	y_3	y_4
x_1	0.88	1	0.9	0.7
x_2	0.9	0.8	0.5	0.7
x_3	0.9	1	0.8	0.8

2.3. Projection

Consider R = "x is more muttaqi than y":

	y_1	y_2	y_3	y_4
x_1	0.8	1	0.1	0.7
x_2	0	0.8	0	0
x_3	0.9	1	0.7	0.8

then the projection of R on X means that x_1 is assigned the highest degree of membership from the tuples (x_1, y_1), (x_1, y_2), (x_1, y_3) and (x_1, y_4), i.e., 1 which is the maximum of the first row.

x_2 is assigned the highest degree of membership from the tuples (x_2, y_1), (x_2, y_2), (x_2, y_3) and (x_2, y_4), i.e., 0.8 which is the maximum of the second row.

x_3 is assigned the highest degree of membership from the tuples (x_3, y_1), (x_3, y_2), (x_3, y_3) and (x_3, y_4), i.e., 1 which is the maximum of the third row.

So one obtains the projection of R on X as

$$\text{Proj R on X} = 1/x_1 + 0.8/x_2 + 1/x_3$$

In the same way, the projection on Y can be taken by searching for the maxima of the four columns. This gives the fuzzy set

$$\text{Proj R on Y} = 0.9/y_1 + 1/y_2 + 0.7/y_3 + 0.8/y_4$$

2.4. Cylindrical Extension

The projection operation brings a ternary relation back to a binary relation, or a binary relation to a fuzzy set, or a fuzzy set to a single crisp value. The projection operation is almost always used in combination with the cylindrical extension. The cylindrical extension is more or less the opposite of the projection. It extends fuzzy sets to fuzzy binary relations, fuzzy binary relations to fuzzy ternary relations and so on.

Cylindrical extension may be easily understood by the following example:

Let the fuzzy set

$$\text{Proj R on X} = 1/x_1 + 0.8/x_2 + 1/x_3 = A$$

then the cylindrical extension of A on the domain $X \times Y$ is given by

$$Ce(A) = \begin{array}{c|cccc|} & y_1 & y_2 & y_3 & y_4 \\ \hline x_1 & 1 & 1 & 1 & 1 \\ x_2 & 0.8 & 0.8 & 0.8 & 0.8 \\ x_3 & 1 & 1 & 1 & 1 \\ \hline \end{array}$$

Consider the fuzzy set

$$\text{Proj R on Y} = 0.9/y_1 + 1/y_2 + 0.7/y_3 + 0.8/y_4 = B$$

then the cylindrical extension of B on the domain $X \times Y$ is given by

Ce(B) =

	y_1	y_2	y_3	y_4
x_1	0.9	1	0.7	0.8
x_2	0.9	1	0.7	0.8
x_3	0.9	1	0.7	0.8

2.5. Composition

Let R be the relation

R = "x is more muttaqī than y"

or R =

	y_1	y_2	y_3	y_4
x_1	0.8	1	0.1	0.7
x_2	0	0.8	0	0
x_3	0.9	1	0.7	0.8

and suppose it is known that "x is momin", which can be expressed by the fuzzy set

$$A = 0.3/x_1 + 1/x_2 + 0.8/x_3$$

The combination of the fuzzy relation R and the fuzzy set A, expressed by "x is more muttaqī than y" and "x is small" can be given by the intersection of the relation and the cylindrical extension of A. The cylindrical extension of A into $X \times Y$ is

Ce(A) =

	y_1	y_2	y_3	y_4
x_1	0.3	0.3	0.3	0.3
x_2	1	1	1	1
x_3	0.8	0.8	0.8	0.8

The intersection of R and Ce(A) is

$R \cap Ce(A)$ =

	y_1	y_2	y_3	y_4
x_1	0.3	0.3	0.1	0.3
x_2	0	0.8	0	0
x_3	0.8	0.8	0.7	0.8

The intersection of fuzzy set and fuzzy relation with the aid of cylindrical extension and then the projection of that intersection is known as composition. The composition operation is denoted by o. It is a special kind of fuzzy intersection.

Let A be a fuzzy set defined on X and R be a fuzzy relation defined on X × Y, then the composition of A and R defined on Y, resulting in a fuzzy set B, is given by

$$B = A \circ R = \text{Proj}\ (Ce(A) \cap R) \text{ on } Y$$

e.g., consider $R \cap Ce(A)$ given in the previous topic then the composition of A and R, resulting in B is given by

$$B = 0.8/y_1 + 0.8/y_2 + 0.7/y_3 + 0.8/y_4$$

Similarly if C is a fuzzy set defined on Y and R is a fuzzy relation defined on X × Y, then the composition of C and R defined on X, resulting in a fuzzy set D, is given by

$$D = C \circ R = \text{Proj}\ (Ce(C) \cap R) \text{ on } X$$

Let "y is sāleh" is expressed by the fuzzy set

$$C = 0.9/y_1 + 1/y_2 + 0.7/y_3 + 0.8/y_4$$

and R =

	y_1	y_2	y_3	y_4
x_1	0.8	1	0.1	0.7
x_2	0	0.8	0	0
x_3	0.9	1	0.7	0.8

Now

Ce(C) =

	y_1	y_2	y_3	y_4
x_1	0.9	1	0.7	0.8
x_2	0.9	1	0.7	0.8
x_3	0.9	1	0.7	0.8

and $R \cap Ce(C)$ =

	y_1	y_2	y_3	y_4
x_1	0.8	1	0.1	0.7
x_2	0	0.8	0	0
x_3	0.9	1	0.7	0.8

And therefore

$$D = C \text{ o } R = \text{Proj } (R \cap Ce(C)) \text{ on } X$$

$$= 1/x_1 + 0.8/x_2 + 1/x_3$$

Suppose there are two relations R and S, where R is defined on X × Y and S is defined on Y × Z. It is of course not possible to take the intersection of R and S, because they are defined on different domains. In this case, one has to extend both relations to X × Y × Z. When this has happened, one can take the intersection. This intersection has to be Projected onto X × Z. Formally T the intersection of R and S is

$$T = R \text{ o } S = \text{Projection of } (Ce(R) \cap Ce(S)) \text{ on } X \times Z$$

i.e. we have to find the composition of R and S

Compositions of binary fuzzy relations can be performed conveniently in terms of membership matrices of the relations. Let $R = [r_{ik}]_{m \times p}$, $S = [s_{kj}]_{p \times n}$ and $T = [t_{ij}]_{m \times n}$ be membership matrices of binary relations such that $T = R \text{ o } S$. We can then write, using this matrix notation,

$$[t_{ij}]_{m \times n} = [r_{ik}]_{m \times p} \text{ o } [s_{kj}]_{p \times n}$$

where $t_{ij} = \max [\min(r_{i1}, s_{1j}), \min(r_{i2}, s_{2j}), \ldots, \min(r_{ip}, s_{pj})]$

or $t_{ij} = \max\limits_{k} \min(r_{ik}, s_{kj})$

Let R = "x is more muttaqi than y":

	y_1	y_2	y_3	y_4
x_1	0.8	1	0.1	0.7
x_2	0	0.8	0	0
x_3	0.9	1	0.7	0.8

and S = "y is closer to Allāh than z":

	z_1	z_2	z_3
y_1	0.4	0.9	0.3
y_2	0	0.4	0
y_3	0.9	0.5	0.8
y_4	0.6	0.7	0.5

then T, the intersection of R and S can be determined as:

t_{11} = max [min(r_{11}, s_{11}), min(r_{12}, s_{21}), min(r_{13}, s_{31}), min(r_{14}, s_{41})]
= max [min(0.8, 0.4), min(1, 0), min(0.1, 0.9), min(0.7, 0.6)]
= max [0.4, 0, 0.1, 0.6] = 0.6

t_{12} = max [min(r_{11}, s_{12}), min(r_{12}, s_{22}), min(r_{13}, s_{32}), min(r_{14}, s_{42})]
=max [min(0.8, 0.9),min(1, 0.4), min(0.1, 0.5), min(0.7, 0.7)]
= max [0.8, 0.4, 0.1, 0.7] = 0.8

t_{13} = max [min(r_{11}, s_{13}), min(r_{12}, s_{23}), min(r_{13}, s_{33}), min(r_{14}, s_{43})]
= max [min(0.8, 0.3), min(1, 0), min(0.1, 0.8), min(0.7, 0.5)]
= max[0.3, 0, 0.1, 0.5] = 0.5

t_{21} = max [min(r_{21}, s_{11}), min(r_{22}, s_{21}), min(r_{23}, s_{31}), min(r_{24}, s_{41})]
= max [min(0, 0.4), min(0.8, 0), min(0, 0.9), min(0, 0.6)]
= max [0, 0, 0, 0] = 0

t_{22} = max [min(r_{21}, s_{12}), min(r_{22}, s_{22}), min(r_{23}, s_{32}), min(r_{24}, s_{42})]
= max [min(0, 0.9), min(0.8, 0.4), min(0, 0.5), min(0, 0.7)]
= max [0, 0.4, 0, 0] = 0.4

t_{23} = max [min(r_{21}, s_{13}), min(r_{22}, s_{23}), min(r_{23}, s_{33}), min(r_{24}, s_{43})]
= max [min(0, 0.3), min(0.8, 0), min(0, 0.8), min(0, 0.5)]
= max [0, 0, 0, 0] = 0

t_{31} = max [min(r_{31}, s_{11}), min(r_{32}, s_{21}), min(r_{33}, s_{31}), min(r_{34}, s_{41})]
= max [min(0.9, 0.4), min(1, 0), min(0.7, 0.9), min(0.8, 0.6)]
= max [0.4, 0, 0.7, 0.6] = 0.7

t_{32} = max [min(r_{31}, s_{12}), min(r_{32}, s_{22}), min(r_{33}, s_{32}), min(r_{34}, s_{42})]

$=\max[\min(0.9, 0.9), \min(1, 0.4), \min(0.7, 0.5), \min(0.8, 0.7)]$
$= \max[0.9, 0.4, 0.5, 0.7] = 0.9$

$t_{33} = \max[\min(r_{31}, s_{13}), \min(r_{32}, s_{23}), \min(r_{33}, s_{33}), \min(r_{34}, s_{43})]$
$= \max[\min(0.9, 0.3), \min(1, 0), \min(0.7, 0.8), \min(0.8, 0.5)]$
$= \max[0.3, 0, 0.7, 0.5] = 0.7$

therefore the intersection

T =

	z_1	z_2	z_3
x_1	0.6	0.8	0.5
x_2	0	0.4	0
x_3	0.7	0.9	0.7

3. FUZZY STATEMENT

A sentence, whose truth value lies in a closed interval [0, 1], is called a fuzzy statement. The truth t, defines a mapping from the set of statements to the set of truth values:

$$t: \{\text{Statement}\} \rightarrow \{\text{Truth Values}\}$$

In classical or Aristotelian logic, truth allows only the two truth values 1 (for true) and 0 (for false), so according to the Classical or Aristotelian logic

$$t: \{\text{Statement}\} \rightarrow \{0, 1\}$$

According to fuzzy logic, on the other hand,

$$t: \{\text{Statement}\} \rightarrow [0, 1]$$

e.g.,

t { Fārābī is muttaqī) = 0.6 and
t (Abu-Talhā is a waliullāh) = 0.7

are the two fuzzy statements, whose truth values $0.6, 0.7 \in [0, 1]$.

3.1. Negation of a Fuzzy Statement

The negation of a fuzzy statement is formed by placing the word "not" within the original or the given statement. The truth value of the negation can be obtained by a complement function like, '1 – a'

where a is the truth value of the original statement e.g., "t (Fārābī is not muttaqī) = 0.4" is the negation of the statement "t(Fārābī is muttaqī) = 0.6"

3.2. Fuzzy Compound Statement

Fuzzy Compound statements are composed of sub statements and various connectives (~, $\wedge$, $\vee$ etc). The fundamental property of a compound statement is that its truth value is completely determined by the way in which its sub statements are connected to form the compound statement.

3.2.1. Fuzzy Conjunction

A fuzzy conjunction is a fuzzy compound statement formed by combining two simple fuzzy statements using the word "and". Let

p: Fārābī is muttaqī with t(p) = 0.6
q: Abu-Talhā is a waliullāh with t(q) = 0.7

then the conjunction is given by the following with the truth value that can be obtained using an intersection function like 'min(a, b)' where a and b are the truth values of the two sub statements p and q.

$t(p \wedge q)$ = t (Fārābī is muttaqī and Abu-Talhā is a waliullāh) = 0.6

3.2.2. Fuzzy Disjunction

A fuzzy disjunction is a fuzzy compound statement formed by combining two simple fuzzy statements using the word "or". If

p: Fārābī is muttaqī with t(p) = 0.6
q: Abu-Talhā is a waliullāh with t(q) = 0.7

then the disjunction is given by the following with the truth value that can be obtained using a union function like 'max(a, b)' where a and b are the truth values of the two sub statements p and q.

$t(p \vee q)$ = t(Fārābī is muttaqī or Abu-Talhā is a waliullāh) = 0.7

4. FUZZY PROPOSITION

A fuzzy compound statement P(p, q, r,....) of the sub fuzzy statements p, q, r, ...

is known as fuzzy proposition, if the sub statements are variables. e.g.

p: X is A (where A is a claim and X is an object and both are variables)
q: Y is B (where Y and B are variables)

These fuzzy atomic propositions p and q can be connected with various connectives like
$\sim, \vee, \wedge, \rightarrow$ etc to form a fuzzy compound proposition.

A conditional fuzzy proposition is expressed as

If X is A then Y is B

where X is A and Y is B are the fuzzy propositions. Each of X is A and Y is B is either an atomic or a compound fuzzy proposition. A conditional fuzzy proposition can be denoted by

X is A $\rightarrow$ Y is B

If each of X is A and Y is B is replaced by a fuzzy statement then a conditional fuzzy statement is obtained, e.g.,

"If Fārābī is muttaqī then he is closed to Allāh"
is a conditional fuzzy statement.

The meaning of Conditional fuzzy proposition and each of its components are given as:

(i) The meaning of "X is A", called the "rule antecedent", is represented by a fuzzy set

$$\tilde{A} = \int_x \mu_A(x)/x$$

(ii) The meaning of "Y is B", called the "rule consequent", is represented by a fuzzy set

$$\tilde{B} = \int_y \mu_B(y)/y$$

(iii) The meaning of the fuzzy conditional is then a fuzzy relation μ_R such that

$$\forall x \in \chi\, \forall y \in Y: \mu_R(x, y) = \mu_A(x) * \mu_B(y)$$

where χ and Y are the domains of X and Y, and $*$ is any fuzzy implication operator that will be discussed in the next section.

5. SOME FUZZY IMPLICATIONS

The truth function for conditional can have many forms. In the following, a number of relations that represent fuzzy implications are given

5.1. Kleene-Dienes or Dienes-Rescher Implication

In section 10, the meaning of the fuzzy conditional is given by the relation

$$\forall x \in \chi \ \forall y \in Y: \mu_R(x, y) = \mu_A(x) * \mu_B(y)$$

If the meaning of this relation is considered as "not X is A or Y is B" and the relations fuzzy complement and fuzzy union are taken as "one minus" and "maximum" operations respectively then the meaning of the relation is given as

$$\mu_R(x, y) = \max(1 - \mu_A(x), \mu_B(y))$$

The relation is known as Kleene-Dienes implication or Dienes-Rescher implication. e.g.,
Consider the rule "if X is A then Y is B", where the meanings of X is A and Y is B are given as

$$\tilde{A} = 0.1/x_1 + 0.4/x_2 + 0.7/x_3 + 1/x_4 \text{ and } \tilde{B} = 0.2/y_1 + 0.5/y_2 + 0.9/y_3$$

The complement of $\tilde{A}$ according to "one minus" operation is

$$\tilde{A}' = 0.9/x_1 + 0.6/x_2 + 0.3/x_3 + 0/x_4$$

Now the union of $\tilde{A}'$ and $\tilde{B}$ first requires the extensions of $\tilde{A}$ and $\tilde{B}$ and then "maximum" operation can be used.

$Ce(\tilde{A}') =$

	y_1	y_2	y_3
x_1	0.9	0.9	0.9
x_2	0.6	0.6	0.6
x_3	0.3	0.3	0.3
x_4	0	0	0

and $Ce(\tilde{B}) =$

	y_1	y_2	y_3
x_1	0.2	0.5	0.9
x_2	0.2	0.5	0.9
x_3	0.2	0.5	0.9
x_4	0.2	0.5	0.9

The "maximum" operation gives the conditional

$\therefore \mu_R(x, y) = Ce(\tilde{A}') \cup Ce(\tilde{B}) =$

	y_1	y_2	y_3
x_1	0.9	0.9	0.9
x_2	0.6	0.6	0.9
x_3	0.3	0.5	0.9
x_4	0.2	0.5	0.9

5.2. Lukasiewicz Implication

The lukasiewicz implication is given by the relation

$$\mu_R(x, y) = \min(1, 1 - \mu_A(x) + \mu_B(y))$$

$Ce(\tilde{A}') =$

	y_1	y_2	y_3
x_1	0.9	0.9	0.9
x_2	0.6	0.6	0.6
x_3	0.3	0.3	0.3
x_4	0	0	0

and $Ce(\tilde{B}) =$

	y_1	y_2	y_3
x_1	0.2	0.5	0.9
x_2	0.2	0.5	0.9
x_3	0.2	0.5	0.9
x_4	0.2	0.5	0.9

$\therefore \mu_R(x, y) =$

	y_1	y_2	y_3
x_1	1	1	1
x_2	0.8	1	1
x_3	0.5	0.8	1
x_4	0.2	0.5	0.9

5.3. Zadeh Implication

The zadeh implication is given by the relation

$$\mu_R(x, y) = \max(\min(\mu_A(x), \mu_B(y)), 1 - \mu_A(x))$$

Now

$$Ce(\tilde{A}') \cap Ce(\tilde{B}) = \begin{array}{c|ccc} & y_1 & y_2 & y_3 \\ \hline x_1 & 0.1 & 0.1 & 0.1 \\ x_2 & 0.4 & 0.4 & 0.4 \\ x_3 & 0.7 & 0.7 & 0.7 \\ x_4 & 1 & 1 & 1 \end{array} \cap \begin{array}{c|ccc} & y_1 & y_2 & y_3 \\ \hline x_1 & 0.2 & 0.5 & 0.9 \\ x_2 & 0.2 & 0.5 & 0.9 \\ x_3 & 0.2 & 0.5 & 0.9 \\ x_4 & 0.2 & 0.5 & 0.9 \end{array}$$

$$= \begin{array}{c|ccc} & y_1 & y_2 & y_3 \\ \hline x_1 & 0.1 & 0.1 & 0.1 \\ x_2 & 0.2 & 0.4 & 0.4 \\ x_3 & 0.2 & 0.5 & 0.7 \\ x_4 & 0.2 & 0.5 & 0.9 \end{array}, \text{ and } Ce(\tilde{A}') = \begin{array}{c|ccc} & y_1 & y_2 & y_3 \\ \hline x_1 & 0.9 & 0.9 & 0.9 \\ x_2 & 0.6 & 0.6 & 0.6 \\ x_3 & 0.3 & 0.3 & 0.3 \\ x_4 & 0 & 0 & 0 \end{array}$$

$$\therefore \mu_R(x, y) = \begin{array}{c|ccc} & y_1 & y_2 & y_3 \\ \hline x_1 & 0.9 & 0.9 & 0.9 \\ x_2 & 0.6 & 0.6 & 0.6 \\ x_3 & 0.3 & 0.5 & 0.7 \\ x_4 & 0.2 & 0.5 & 0.9 \end{array}$$

5.4. Stochastic Implication

The stochastic implication is given by the relation

$$\mu_R(x, y) = \max(1 - \mu_A(x), \mu_A(x).\mu_B(x))$$

$$Ce(\tilde{A}') = \begin{array}{c|ccc} & y_1 & y_2 & y_3 \\ \hline x_1 & 0.9 & 0.9 & 0.9 \\ x_2 & 0.6 & 0.6 & 0.6 \\ x_3 & 0.3 & 0.3 & 0.3 \\ x_4 & 0 & 0 & 0 \end{array}$$

$$
Ce(\tilde{A}') \cap Ce(\tilde{B}) = \begin{array}{c|ccc} & y_1 & y_2 & y_3 \\ \hline x_1 & 0.02 & 0.05 & 0.09 \\ x_2 & 0.08 & 0.2 & 0.36 \\ x_3 & 0.14 & 0.35 & 0.63 \\ x_4 & 0.2 & 0.5 & 0.9 \end{array}
$$

$$
\therefore \mu_R(x, y) = \begin{array}{c|ccc} & y_1 & y_2 & y_3 \\ \hline x_1 & 0.9 & 0.9 & 0.9 \\ x_2 & 0.6 & 0.6 & 0.6 \\ x_3 & 0.3 & 0.35 & 0.63 \\ x_4 & 0.2 & 0.5 & 0.9 \end{array}
$$

5.5. Goguen Implication

The goguen implication is given by

$$\mu_R(x, y) = \min(1, \mu_A(x) / \mu_B(y))$$

Applying min between

$$
\begin{array}{c|ccc} & y_1 & y_2 & y_3 \\ \hline x_1 & 1 & 1 & 1 \\ x_2 & 1 & 1 & 1 \\ x_3 & 1 & 1 & 1 \\ x_4 & 1 & 1 & 1 \end{array}
\quad \text{and} \quad
\begin{array}{c|ccc} & y_1 & y_2 & y_3 \\ \hline x_1 & 0.5 & 0.2 & 0.11 \\ x_2 & 2 & 0.8 & 0.44 \\ x_3 & 3.5 & 1.4 & 0.77 \\ x_4 & 5 & 2 & 1.11 \end{array}
$$

we get

$$
\therefore \mu_R(x, y) = \begin{array}{c|ccc} & y_1 & y_2 & y_3 \\ \hline x_1 & 0.5 & 0.2 & 0.11 \\ x_2 & 1 & 0.8 & 0.44 \\ x_3 & 1 & 1 & 0.77 \\ x_4 & 1 & 1 & 1 \end{array}
$$

5.6. Gödel Implication

The Gödel implication is given by

$$\mu_R(x, y) = \begin{cases} 1, & \text{if } \mu_A(x) \le \mu_B(y) \\ \mu_B(y), & \text{otherwise} \end{cases}$$

For

$$Ce(\tilde{A}) = \begin{array}{c|ccc|} & y_1 & y_2 & y_3 \\ x_1 & 0.1 & 0.1 & 0.1 \\ x_2 & 0.4 & 0.4 & 0.4 \\ x_3 & 0.7 & 0.7 & 0.7 \\ x_4 & 1 & 1 & 1 \end{array} \quad \text{and} \quad Ce(\tilde{B}) = \begin{array}{c|ccc|} & y_1 & y_2 & y_3 \\ x_1 & 0.2 & 0.5 & 0.9 \\ x_2 & 0.2 & 0.5 & 0.9 \\ x_3 & 0.2 & 0.5 & 0.9 \\ x_4 & 0.2 & 0.5 & 0.9 \end{array}$$

$$\therefore \mu_R(x, y) = \begin{array}{c|ccc|} & y_1 & y_2 & y_3 \\ x_1 & 1 & 1 & 1 \\ x_2 & 0.2 & 1 & 1 \\ x_3 & 0.2 & 0.5 & 1 \\ x_4 & 0.2 & 0.5 & 0.9 \end{array}$$

5.7. Sharp Implication

The sharp implication is given by

$$\mu_R(x, y) = \begin{cases} 1, & \text{if } \mu_A(x) \le \mu_B(y) \\ 0, & \text{otherwise} \end{cases}$$

For

$$Ce(\tilde{A}) = \begin{array}{c|ccc} & y_1 & y_2 & y_3 \\ \hline x_1 & 0.1 & 0.1 & 0.1 \\ x_2 & 0.4 & 0.4 & 0.4 \\ x_3 & 0.7 & 0.7 & 0.7 \\ x_4 & 1 & 1 & 1 \end{array} \quad \text{and} \quad Ce(\tilde{B}) = \begin{array}{c|ccc} & y_1 & y_2 & y_3 \\ \hline x_1 & 0.2 & 0.5 & 0.9 \\ x_2 & 0.2 & 0.5 & 0.9 \\ x_3 & 0.2 & 0.5 & 0.9 \\ x_4 & 0.2 & 0.5 & 0.9 \end{array}$$

$$\therefore \mu_R(x, y) = \begin{array}{c|ccc} & y_1 & y_2 & y_3 \\ \hline x_1 & 1 & 1 & 1 \\ x_2 & 0 & 1 & 1 \\ x_3 & 0 & 0 & 1 \\ x_4 & 0 & 0 & 0 \end{array}$$

5.8. Mamdani Implication

The Mamdani implication is given by the relation

$$\mu_R(x, y) = \min(\mu_A(x), \mu_B(y))$$

For

$$Ce(\tilde{A}) = \begin{array}{c|ccc} & y_1 & y_2 & y_3 \\ \hline x_1 & 0.1 & 0.1 & 0.1 \\ x_2 & 0.4 & 0.4 & 0.4 \\ x_3 & 0.7 & 0.7 & 0.7 \\ x_4 & 1 & 1 & 1 \end{array} \quad \text{and} \quad Ce(\tilde{B}) = \begin{array}{c|ccc} & y_1 & y_2 & y_3 \\ \hline x_1 & 0.2 & 0.5 & 0.9 \\ x_2 & 0.2 & 0.5 & 0.9 \\ x_3 & 0.2 & 0.5 & 0.9 \\ x_4 & 0.2 & 0.5 & 0.9 \end{array}$$

$$\therefore \mu_R(x, y) = \begin{array}{c|ccc} & y_1 & y_2 & y_3 \\ \hline x_1 & 0.1 & 0.1 & 0.1 \\ x_2 & 0.2 & 0.4 & 0.4 \\ x_3 & 0.2 & 0.5 & 0.7 \\ x_4 & 0.2 & 0.5 & 0.9 \end{array}$$

6. BI-CONDITIONAL FUZZY PROPOSITION AND LOGICAL EQUIVALENCE

Statements A and B are logically equivalent or A=B if and only if A implies B and B implies A. The Lukasiewics equivalence has the form

$$t_L(A = B) = 1 - |t(A) - t(B)|$$

So two fuzzy statements A and B are (100%) equivalent if and only if they have the same truth values:

$$t(A) = t(B)$$

7. FUZZY ARGUMENT

Inference rules in classical logic are based on the various tautologies. These are *Modus Ponens, Modus Tollens, Hypothetical Syllogism, Disjunctive Syllogism, Constructive Dilemma, Absorption, Simplification, Conjunction* and *Addition.* These tautologies are known as "Rules of Inference" and are the elementary valid argument forms, whose validity is easily established by truth tables. They can be used to construct formal proofs of validity for a wide range of more complicated arguments. These inference rules can be generalized within the framework of fuzzy logic to facilitate approximate reasoning, e.g., for the *Generalized Modus Ponens*, we follow the following procedure

Let R is a fuzzy relation in $X \times Y$. Let P and Q are fuzzy sets on X and Y respectively, then if R and P are given, we can compute Q by the equation

$$Q(y) = \sup x \in X \min[P(x), R(x, y)]$$

for all $y \in Y$. This equation can also be written in the matrix notation as follows and is known as "*Compositional Rule of Inference*".

$$Q = P \circ R$$

Let X is A: ruler is momin

Y is B: judiciary is impartial

Z is C: public is happy

where the meanings of "X is A", "Y is B" and "Z is C" are given as

$$\tilde{A} = 0.5/x_1 + 1/x_2 + 0.6/x_3,\ \tilde{B} = 1/y_1 + 0.4/y_2,\ \tilde{C} = 0.2/z_1 + 1/z_2$$

Then the three generalized inference rules, i.e., *Generalized Modus Ponens, Generalized Modus Tollens, and Generalized*

Hypothetical Syllogism, based on the "*Compositional Rule of Inference*", are described below:

7.1. Generalized Modus Ponens

The argument of the type "If X is A then Y is B, X is P, therefore Y is Q" is in Generalized Modus Ponens form where the meaning of "X is P" is $\tilde{P} = 0.6/x_1 + 0.9/x_2 + 0.7/x_3$. The Generalized Modus Ponens can also be written as:

$$\begin{array}{ll} & \text{If X is A then Y is B} \\ & \underline{\text{X is P}\qquad\qquad\qquad} \\ \text{therefore} & \text{Y is Q} \end{array}$$

Now it is required to compute the conclusion "Y is Q". The procedure of the computation is given below

The cylindrical extensions of $\tilde{A}'$, $\tilde{B}$ and $\tilde{P}$ are

$$Ce(\tilde{A}') = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{cc} y_1 & y_2 \\ \hline \multicolumn{1}{|c|}{0.5} & \multicolumn{1}{c|}{0.5} \\ \multicolumn{1}{|c|}{0} & \multicolumn{1}{c|}{0} \\ \multicolumn{1}{|c|}{0.4} & \multicolumn{1}{c|}{0.4} \\ \hline \end{array}$$

$$Ce(\tilde{B}) = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{cc} y_1 & y_2 \\ \hline \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{0.4} \\ \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{0.4} \\ \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{0.4} \\ \hline \end{array}$$

and

$$Ce(\tilde{P}) = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{cc} y_1 & y_2 \\ \hline \multicolumn{1}{|c|}{0.6} & \multicolumn{1}{c|}{0.6} \\ \multicolumn{1}{|c|}{0.9} & \multicolumn{1}{c|}{0.9} \\ \multicolumn{1}{|c|}{0.7} & \multicolumn{1}{c|}{0.7} \\ \hline \end{array}$$

Using Lukasiewicz implication

$$Ce(\tilde{A}') \vee Ce(\tilde{B}) = \begin{array}{c|cc} & y_1 & y_2 \\ \hline x_1 & 1 & 0.9 \\ x_2 & 1 & 0.4 \\ x_3 & 1 & 0.8 \end{array}$$

$$[Ce(\tilde{A}') \vee Ce(\tilde{B})] \wedge Ce(\tilde{P}) = \begin{array}{c|cc} & y_1 & y_2 \\ \hline x_1 & 1 & 0.9 \\ x_2 & 1 & 0.4 \\ x_3 & 1 & 0.8 \end{array} = Ce(\tilde{Q})$$

The projection of $Ce(\tilde{Q})$ on y gives

$$\tilde{Q} = 0.9/y_1 + 0.7/y_2$$

7.2. Generalized Modus Tollens

The argument of the type "If X is A then Y is B, Y is Q, therefore X is P" is in Generalized Modus Tollens form where the meaning of "Y is Q" is $\tilde{Q} = 0.9/y_1 + 0.7/y_2$. The Generalized Modus Tollens can also be written as:

If X is A then Y is B
Y is Q
therefore X is P

Now it is required to compute the conclusion "X is P". The procedure of the computation is given below

The cylindrical extensions of $\tilde{A}'$, $\tilde{B}$ and $\tilde{Q}$ are

$$Ce(\tilde{A}') = \begin{array}{c|cc} & y_1 & y_2 \\ \hline x_1 & 0.5 & 0.5 \\ x_2 & 0 & 0 \\ x_3 & 0.4 & 0.4 \end{array}$$

$$Ce(\bar{B}) = \begin{array}{c|cc} & y_1 & y_2 \\ \hline x_1 & 1 & 0.4 \\ x_2 & 1 & 0.4 \\ x_3 & 1 & 0.4 \end{array}$$

and

$$Ce(\bar{Q}) = \begin{array}{c|cc} & y_1 & y_2 \\ \hline x_1 & 0.9 & 0.7 \\ x_2 & 0.9 & 0.7 \\ x_3 & 0.9 & 0.7 \end{array}$$

Using Lukasiewicz implication

$$Ce(\bar{A}') \vee Ce(\bar{B}) = \begin{array}{c|cc} & y_1 & y_2 \\ \hline x_1 & 1 & 0.9 \\ x_2 & 1 & 0.4 \\ x_3 & 1 & 0.8 \end{array}$$

$$[Ce(\bar{A}') \vee Ce(\bar{B})] \wedge Ce(\bar{Q}) = \begin{array}{c|cc} & y_1 & y_2 \\ \hline x_1 & 0.9 & 0.7 \\ x_2 & 0.9 & 0.4 \\ x_3 & 0.9 & 0.7 \end{array} = Ce(\bar{P})$$

The projection of $Ce(\bar{P})$ on x gives

$$\bar{P} = 0.9/x_1 + 0.9/x_2 + 0.9/x_3$$

7.3. Generalized Hypothetical Syllogism

The argument of the type "If X is A then Y is B, if Y is B then Z is C, therefore if X is A then Z is C " is in Generalized Hypothetical Syllogism form. The Generalized Hypothetical Syllogism can also be written as:

	If X is A then Y is B
	if Y is B then Z is C
therefore	if X is A then Z is C

Now it is required to compute $\mu_{R1}(x, y)$, $\mu_{R2}(x, y)$ and $\mu_{R3}(x, y)$. The Generalized Hypothetical Syllogism holds if $\mu_{R3}(x, y) = \mu_{R1}(x, y) \circ \mu_{R2}(x, y)$. The procedure of the computation is given below

The cylindrical extensions of $\tilde{A}$ and $\tilde{B}$ are

$$Ce(\tilde{A}) = \begin{array}{c|c|c|} & y_1 & y_2 \\ \hline x_1 & 0.5 & 0.5 \\ \hline x_2 & 1 & 1 \\ \hline x_3 & 0.6 & 0.6 \\ \hline \end{array}$$

and

$$Ce(\tilde{B}) = \begin{array}{c|c|c|} & y_1 & y_2 \\ \hline x_1 & 1 & 0.4 \\ \hline x_2 & 1 & 0.4 \\ \hline x_3 & 1 & 0.4 \\ \hline \end{array}$$

Using Gödel implication

$$\mu_{R1}(x, y) = \begin{array}{c|c|c|} & y_1 & y_2 \\ \hline x_1 & 1 & 0.4 \\ \hline x_2 & 1 & 0.4 \\ \hline x_3 & 1 & 0.4 \\ \hline \end{array}$$

The cylindrical extensions of $\tilde{B}$ and $\tilde{C}$ are

$$Ce(\tilde{B}) = \begin{array}{c|c|c|} & z_1 & z_2 \\ \hline y_1 & 1 & 1 \\ \hline y_2 & 0.4 & 0.4 \\ \hline \end{array}$$

and

$$Ce(\tilde{C}) = \begin{array}{c|cc} & z_1 & z_2 \\ \hline y_1 & 0.2 & 1 \\ y_2 & 0.2 & 1 \end{array}$$

Using Gödel implication

$$\mu_{R2}(x, y) = \begin{array}{c|cc} & z_1 & z_2 \\ \hline y_1 & 0.2 & 1 \\ y_2 & 0.2 & 1 \end{array}$$

The cylindrical extensions of $\tilde{A}$ and $\tilde{C}$ are

$$Ce(\tilde{A}) = \begin{array}{c|cc} & z_1 & z_2 \\ \hline x_1 & 0.5 & 0.5 \\ x_2 & 1 & 1 \\ x_3 & 0.6 & 0.6 \end{array}$$

and

$$Ce(\tilde{C}) = \begin{array}{c|cc} & z_1 & z_2 \\ \hline x_1 & 1 & 0.4 \\ x_2 & 1 & 0.4 \\ x_3 & 1 & 0.4 \end{array}$$

Using Gödel implication

$$\mu_{R3}(x, y) = \begin{array}{c|cc} & z_1 & z_2 \\ \hline x_1 & 0.2 & 1 \\ x_2 & 0.2 & 1 \\ x_3 & 0.2 & 1 \end{array}$$

Now $\mu_{R1}(x, y) \circ \mu_{R2}(x, y) = \begin{array}{c|cc} & z_1 & z_2 \\ \hline x_1 & 0.2 & 1 \\ x_2 & 0.2 & 1 \\ x_3 & 0.2 & 1 \end{array} = \mu_{R3}(x, y)$

Therefore the Generalized Hypothetical Syllogism holds

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